Journal of Algebra 357 (2012) 263–270

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Finite p-groups with many minimal nonabelian subgroups

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ARTICLE INFO

Article history:
Received 20 October 2011
Available online 2 March 2012
Communicated by E.I. Khukhro

MSC:
20D15

Keywords:
Minimal nonabelian p-groups
Metacyclic p-groups
Ordinary metacyclic 2-groups
A Suzuki 2-group of order 2^6

ABSTRACT

In Theorem 2.1 we characterize finite p-groups G such that each nonabelian subgroup H of G which possesses an abelian maximal subgroup is minimal nonabelian. In Theorem 3.1 the problem 2331 of Y. Berkovich (stated in Y. Berkovich and Z. Janko, in preparation [4]) about finite p-groups with "many" minimal nonabelian subgroups is solved.

The purpose of this paper is to demonstrate how the appearance of many minimal nonabelian subgroups in finite p-groups can influence the structure of such p-groups. We shall consider here only finite p-groups and our notation is standard (see [1]). In particular, for a p-group G, $\mathrm{d}(G)$ denotes the minimal number of generators and $\mathrm{K}_i(G)$ is the i-th member of the lower central series of G, where $\mathrm{K}_1(G) = G$. Also, E_{p^n} denotes the elementary abelian group of order p^n and C_n is the cyclic group of order n. We recall that a metacyclic 2-group G is called "ordinary metacyclic" (with respect to a subgroup H) if G has a cyclic normal subgroup H such that G/H is cyclic and G centralizes $H/\mho_2(H)$ (i.e., $[G, H] \leqslant \mho_2(H)$) (see [1, §26, Definition 7]).

A nonabelian p-group all of whose maximal subgroups are abelian is called minimal nonabelian. Such groups are completely determined by L. Rédei (see [2, Lemma 65.1]). Hence p-groups which possess (only) one abelian maximal subgroup present a much larger class of groups. We have noticed (see Proposition 1.10) that metacyclic p-groups G with $p > 2$ have the following property:

($*$) Whenever a nonabelian subgroup H of G has an abelian maximal subgroup, then H is minimal nonabelian, i.e., each maximal subgroup of H is abelian.

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2012.02.014

Therefore it is of interest to classify all p-groups G which have the above property $(*)$. This will be done in Theorem 2.1. It turns out that not all metacyclic 2-groups but only ordinary metacyclic 2-groups have the property $(*)$. There are also some nonmetacyclic p-groups having the property $(*)$.

In Section 3 we shall consider nonabelian p-groups G which are not minimal nonabelian and which possess the following property:

$(**)$ Whenever X and Y are distinct minimal nonabelian subgroups of G and $x \in X \setminus Y$, $y \in Y \setminus X$, then $\langle x, y\rangle$ is also minimal nonabelian.

It turns out (Theorem 3.1) that p-groups G having the property $(**)$ have also the property $(*)$ and we must have $p = 2$ and G is a special group of order 2^6 which is isomorphic to a Sylow 2-subgroup of the Suzuki simple group $\mathrm{Sz}(8)$. It is surprising that in this case our group G is unique!

In any case, the property $(*)$ is much more general than the property $(**)$ and actually both properties $(*)$ and $(**)$ state that in some sense our p-groups have "many" minimal nonabelian subgroups.

One of the most difficult open problems in p-group theory is to classify p-groups which are covered by its minimal nonabelian subgroups. We see that p-groups with the property $(*)$ which are studied in this paper are in fact covered by its minimal nonabelian subgroups. This also indicates that the above open problem is really difficult.

1. Preliminary results

Proposition 1.1. *(See [2, Lemma 65.1].) Let G be a minimal nonabelian p-group. Then we have $\mathrm{d}(G) = 2$, $|G'| = p$, $\Phi(G) = \mathrm{Z}(G)$ and G is nonmetacyclic if and only if G' is a maximal cyclic subgroup of G. If $|G| > p^3$, then G is nonmetacyclic if and only if $\Omega_1(G) \cong \mathrm{E}_{p^3}$.*

Proposition 1.2. *(See [2, Lemma 65.2(a)].) Let G be a p-group with $\mathrm{d}(G) = 2$ and $|G'| = p$. Then G is minimal nonabelian.*

Proposition 1.3. *(See [3, prerequisites, Exercise P1].) Let G be a nonabelian p-group with two distinct abelian maximal subgroups. Then we have $|G'| = p$.*

Proposition 1.4. *(See [2, Theorem 66.1 and Theorem 69.1].) Let G be a minimal nonmetacyclic p-group. Then one of the following holds*:

(a) *G is of order p^3 and exponent p*;
(b) *G is of maximal class and order 3^4*;
(c) *$p = 2$ and either $|G| = 2^4$ (with $G \cong \mathrm{Q}_8 \times \mathrm{C}_2$ or $G \cong \mathrm{Q}_8 * \mathrm{C}_4$) or G is a special group of order 2^5 and exponent 4 with $\Omega_1(G) = G' = \mathrm{Z}(G) = \Phi(G) \cong \mathrm{E}_4$, and G has a unique abelian maximal subgroup (of type $(4, 4)$).*

Proposition 1.5. *Let G be a p-group of order p^4. Then G has an abelian maximal subgroup.*

Proof. Let U be a normal subgroup of order p^2 in G. Then $\mathrm{C}_G(U) > U$ and so $\mathrm{C}_G(U)$ contains an abelian subgroup of order p^3 and we are done. □

Proposition 1.6. *(See [2, Proposition 71.5].) Let G be a nonmetacyclic p-group of order $> p^4$ all of whose maximal subgroups are minimal nonabelian. Then we have one of the following possibilities*:

(a) $\mathrm{d}(G) = 3$, $p = 2$, *and*

$$G = \langle a, b, c \mid a^4 = b^4 = c^4 = 1,\ [a, b] = c^2,\ [a, c] = b^2c^2,\ [b, c] = a^2b^2,$$
$$[a^2, b] = [a^2, c] = [b^2, a] = [b^2, c] = [c^2, a] = [c^2, b] = 1\rangle,$$

where $|G| = 2^6$,

$$G' = \langle a^2, b^2, c^2 \rangle = Z(G) = \Phi(G) = \Omega_1(G) \cong E_8,$$

G is of exponent 4 *and G is isomorphic to a Sylow* 2*-subgroup of the simple group* Sz(8)*. Each maximal subgroup of G is isomorphic to the minimal nonabelian group*

$$\mathcal{H}_{32} = \langle x, y \mid x^4 = y^4 = 1,\ [x, y] = z,\ z^2 = [z, x] = [z, y] = 1 \rangle$$

of order 2^5*.*

(b) $d(G) = 2$, $p > 2$, *and G is one of the following Blackburn groups*:

$$G = \langle a, x \mid a^{p^2} = x^{p^2} = 1,\ [a, x] = b,\ [a, b] = y_1,\ [x, b] = y_2,\ a^p = y_1^{\alpha} y_2^{\beta},\ x^p = y_1^{\gamma} y_2^{\delta},$$
$$b^p = y_1^p = y_2^p = [a, y_1] = [x, y_1] = [a, y_2] = [x, y_2] = 1 \rangle,$$

where in case $p > 3$, $4\beta\gamma + (\delta - \alpha)^2$ *is a quadratic non-residue mod p. Here* $|G| = p^5$*, and*

$$\Phi(G) = G' = \langle b, y_1, y_2 \rangle = \Omega_1(G) \cong E_{p^3}, \qquad Z(G) = K_3(G) = \mho_1(G) = \langle y_1, y_2 \rangle \cong E_{p^2}.$$

Proposition 1.7. *(See [2, Theorem 57.5].) Let G be a nonabelian* 2*-group all of whose minimal nonabelian subgroups are isomorphic to the following group of order* 2^5:

$$\mathcal{H}_{32} = \langle x, y \mid x^4 = y^4 = 1,\ [x, y] = z,\ z^2 = [z, x] = [z, y] = 1 \rangle.$$

Then $\Omega_1(G) \leqslant Z(G)$ *and G is of exponent* 4 *and class* 2*.*

Proposition 1.8. *(See [1, Corollary 10.6].) Let G be a p-group with* $p > 2$*. Suppose that G has no normal elementary abelian subgroup of order* p^3*. Then G has no elementary abelian subgroup of order* p^3*.*

Proposition 1.9. *(See [3, Lemma 139.1].) Let* $G = \langle x, y \rangle$ *be a nonabelian two-generator p-group with the cyclic commutator group* $G' = \langle [x, y] \rangle$*. If* $p > 2$ *or* $p = 2$ *and* $[G', G] \leqslant \mho_2(G')$*, then we have*

$$\mho_1(G') = \langle [x, y]^p \rangle = \langle [x, y^p] \rangle.$$

Proposition 1.10. *Let G be a nonabelian metacyclic p-group with* $p > 2$*. If G possesses an abelian maximal subgroup A, then G is minimal nonabelian.*

Proof. We may choose generators x, y of G such that $x \in A$ and $y \in G \setminus A$. Then the subgroup $G' = \langle [x, y] \rangle \neq \{1\}$ is cyclic. By Proposition 1.9, we get $\mho_1(G') = \langle [x, y]^p \rangle = \langle [x, y^p] \rangle$ and so $\mho_1(G') = \{1\}$ since $x, y^p \in A$ and so $[x, y^p] = 1$. Hence we have $d(G) = 2$ and $|G'| = p$ so that Proposition 1.2 implies that G is minimal nonabelian. □

We recall that a metacyclic 2-group G is called “ordinary metacyclic” (with respect to a subgroup H) if G has a cyclic normal subgroup H such that G/H is cyclic and G centralizes $H/\mho_2(H)$ (i.e., $[G, H] \leqslant \mho_2(H)$) (see [1, §26, Definition 7]).

Proposition 1.11. *Let G be a nonabelian ordinary metacyclic* 2*-group with respect to a subgroup H. Then we have* $[G', G] \leqslant \mho_2(G')$ *and each subgroup of G is also ordinary metacyclic. If G has an abelian maximal subgroup, then G is minimal nonabelian.*

Proof. Set $H = \langle h \rangle$ and let $g \in G \setminus H$ be such that $\langle g \rangle$ covers G/H and so $G = \langle g, h \rangle$. Because $[G, H] \leqslant \mho_2(H) = \langle h^4 \rangle$, we have $h^g = hv$ with $v \in \langle h^4 \rangle$ and so $[h, g] = v$, $G' = \langle v \rangle$ and $v = h^i$ with $i \equiv 0 \pmod 4$. We compute

$$v^g = \left(h^i\right)^g = \left(h^g\right)^i = (hv)^i = h^i v^i = v v^i,$$

and so $[v, g] = v^i \in \mho_2(G')$ since $i \equiv 0 \pmod 4$. We have proved that $[G', G] \leqslant \mho_2(G')$.

Let U be any subgroup of G and set $V = U \cap H$ so that V is a cyclic normal subgroup of G and U/V is cyclic. We have $V = \langle h^j \rangle$ for some integer j. We compute

$$\left(h^j\right)^g = \left(h^g\right)^j = (hv)^j = h^j v^j = h^j \left(h^i\right)^j = h^j \left(h^j\right)^i,$$

and so $[h^j, g] = (h^j)^i \in \mho_2(V)$, where $i \equiv 0 \pmod 4$. It follows that g centralizes $V/\mho_2(V)$ and so U centralizes $V/\mho_2(V)$ and so U is ordinary metacyclic.

Suppose that G has an abelian maximal subgroup A. We may choose generators x, y of G such that $x \in A$ and $y \in G \setminus A$. Then the subgroup $G' = \langle [x, y] \rangle \neq \{1\}$ is cyclic. Since $[G', G] \leqslant \mho_2(G')$, we may use Proposition 1.9 and we get

$$\mho_1\left(G'\right) = \left\langle [x, y]^2 \right\rangle = \left\langle \left[x, y^2\right] \right\rangle = \{1\},$$

since $x, y^2 \in A$ and so $[x, y^2] = 1$. Hence we have $\mathrm{d}(G) = 2$ and $|G'| = 2$ so that Proposition 1.2 implies that G is minimal nonabelian. Our proposition is proved. □

2. Proof of the main result

Theorem 2.1. *Let G be a nonabelian p-group which is not minimal nonabelian. Suppose that G has the following property*:

$(*)$ *Whenever a nonabelian subgroup H of G has an abelian maximal subgroup, then H is minimal nonabelian.*

Then we have one of the following possibilities:

(a) *G is any of the nonmetacyclic p-groups defined in Proposition* 1.6 (a) *and* (b).
(b) *G is a metacyclic p-group with $p > 2$ and $|G'| > p$.*
(c) *G is an ordinary metacyclic 2-group with $|G'| > 2$.*

Conversely, each of the groups given in (a), (b) *and* (c) *satisfy the assumptions of our theorem.*

Proof. Let G be a nonabelian p-group which is not minimal nonabelian and assume that G has the property $(*)$. This implies that G has no abelian maximal subgroup.

(i) *First assume that G possesses a normal elementary abelian subgroup E of order p^3.*

Let A be a maximal normal abelian subgroup of G which contains E so that A is of rank $\geqslant 3$. Since G has no abelian maximal subgroup, we have $|G/A| > p$. We consider a subgroup H/A of order p^2 in G/A. Let H_1/A be a subgroup of order p in H/A so that (by the property $(*)$) H_1 is minimal nonabelian. Then we have $\Phi(H_1) < A$ and $|A : \Phi(H_1)| = p$ which together with $\Phi(H_1) \leqslant \Phi(H)$ gives $\mathrm{d}(H) \leqslant 3$. If $\mathrm{d}(H) = 3$, then we have $\Phi(H_1) = \Phi(H)$ which implies $H/A \cong \mathrm{E}_{p^2}$. Now suppose that $\mathrm{d}(H) = 2$. In that case we have $\Phi(H) < H_1$ so that $\Phi(H)$ is abelian and $H/\Phi(H) \cong \mathrm{E}_{p^2}$. Note that H_1 is minimal nonabelian and $H_1 < H$ and so H is neither abelian nor minimal nonabelian. It follows by the property $(*)$ that each maximal subgroup of H is minimal nonabelian. We have $E \leqslant A$ and so $|H| > p^4$ and H is nonmetacyclic so that we may use Proposition 1.6(b). It follows that $E = \Omega_1(H) = \Phi(H)$, $H/E \cong \mathrm{E}_{p^2}$ and $A = E$. In particular, we get again $H/A \cong \mathrm{E}_{p^2}$.

We have proved that for any subgroup H/A of order p^2 in G/A we have $H/A \cong \mathrm{E}_{p^2}$. This gives that G/A is of exponent p. We have either $\Phi(H) = A$ or $\Phi(H) = \Phi(H_i)$, where H_i/A is any subgroup of order p in H/A and (by the property $(*)$) H_i is minimal nonabelian, $i = 1, 2, \dots, p+1$.

Suppose that there is at least one subgroup H/A of order p^2 in G/A such that $\Phi(H) = A$ and so $\mathrm{d}(H) = 2$. Since each maximal subgroup H_i of H is nonabelian and $|H_i : A| = p$, it follows by the property $(*)$ that H_i is minimal nonabelian for each $i = 1, 2, \dots, p+1$. Because A is of rank $\geqslant 3$, we have $|H| > p^4$ and H is nonmetacyclic. By Proposition 1.6(b), we get $p > 2$, $|H| = p^5$, $A = H' = \Omega_1(H) = \Phi(H) \cong \mathrm{E}_{p^3}$ and $\mathrm{Z}(H) = \mho_1(H) = \mathrm{K}_3(H)$ is a subgroup of index p in A.

(i1) *Assume that there is at least one subgroup H/A of order p^2 in G/A such that* $\mathrm{d}(H) = 3$.

We know that in that case $\Phi(H) = \Phi(H_i)$, where H_i/A is any subgroup of order p in H/A and (by the property $(*)$) H_i is minimal nonabelian, $i = 1, 2, \dots, p+1$. Also we have $\Phi(H_i) = \mathrm{Z}(H_i)$ for all $i = 1, 2, \dots, p+1$ (Proposition 1.1) which implies $\Phi(H) \leqslant \mathrm{Z}(H)$. On the other hand, H has no abelian maximal subgroup (otherwise, the property $(*)$ would infer that H is minimal nonabelian) and so we get $\Phi(H) = \mathrm{Z}(H)$. Let M be any maximal subgroup of H which does not contain A. Then we have $M \cap A = \Phi(H)$ and so M covers H/A. Let X be a maximal subgroup of M which contains $\Phi(H)$. Since $\Phi(H) = \mathrm{Z}(H)$ and $|X : \Phi(H)| = p$, we see that X is abelian. But M is nonabelian and so, by the property $(*)$, M is minimal nonabelian. Thus H is a nonmetacyclic p-group of order $> p^4$ with $\mathrm{d}(H) = 3$ all of whose maximal subgroups are minimal nonabelian. By Proposition 1.6(a), we get $p = 2$ and H is a special group of order 2^6 which is isomorphic to an S_2-subgroup of the simple group $\mathrm{Sz}(8)$. This gives $\Phi(H) = \mathrm{Z}(H) = H' = \Omega_1(H) = E \cong \mathrm{E}_8$, A is abelian of type $(4, 2, 2)$ and each maximal subgroup of H is isomorphic to $\mathcal{H}_{32}$ defined in Proposition 1.6(a). Since $p = 2$, it follows by the preceding paragraph that for each subgroup K/A of order 4 in G/A we have $\mathrm{d}(K) = 3$ and so, by the above, K is isomorphic to an S_2-subgroup of the simple group $\mathrm{Sz}(8)$. Also, $\exp(G/A) = 2$ gives that G/A is elementary abelian.

Let X be any minimal nonabelian subgroup in G. Since $\mathrm{d}(X) = 2$ (see Proposition 1.1), we have $|X : (X \cap A)| \leqslant 4$ and so X is contained in a subgroup $H > A$ such that $H/A \cong \mathrm{E}_4$. By the above, H is isomorphic to an S_2-subgroup of the simple group $\mathrm{Sz}(8)$ and so (see Proposition 1.6(a)) $X \cong \mathcal{H}_{32}$. By Proposition 1.7, $\Omega_1(G) \leqslant \mathrm{Z}(G)$ and G is of exponent 4 and class 2. Since $\mathrm{C}_G(A) = A$, we have

$$\Omega_1(G) = \Omega_1(A) = E = \mathrm{Z}(G) \cong \mathrm{E}_8.$$

Thus $G' \leqslant E$ and since $H' = E$, we get $G' = E$. In addition (since $\exp(G) = 4$), for each $g \in G \setminus E$, we have $g^2 \in E \setminus \{1\}$ and so $\Phi(G) = E$ and therefore G is a special 2-group.

Let $a \in A \setminus E$ so that $a^2 = z$ is an involution in E and we have $\mho_1(A) = \langle z \rangle$. There are exactly eight elements of order 4 in $A \setminus E$ and $\mathrm{C}_G(A) = A$ implies that $\mathrm{C}_G(a) = A$. This gives that $|G/A| \leqslant 8$. Suppose that $|G/A| = 8$. In that case all eight elements of order 4 in $A \setminus E$ form a single conjugate class in G. In particular, there is an element $g \in G \setminus A$ such that $g^2 \in E$ and $a^g = az = a^{-1}$. By the property $(*)$, $B = A\langle g \rangle$ is minimal nonabelian and so, by the above, $B \cong \mathcal{H}_{32}$ with $B' = \langle z \rangle$. On the other hand, since $\mathcal{H}_{32}$ is nonmetacyclic, it follows (Proposition 1.1) that $B' = \langle z \rangle$ must be a maximal cyclic subgroup in B. This is a contradiction since $z = a^2$. We have proved that $|G/A| = 4$ and so $G = H$ is a special group of order 2^6 which is isomorphic to an S_2-subgroup of the simple group $\mathrm{Sz}(8)$.

(i2) *Now we assume that for all subgroups H/A of order p^2 in G/A we have* $\mathrm{d}(H) = 2$.

Then we already know that we must have $p > 2$ and $|H| = p^5$, where H is isomorphic to a Blackburn group from Proposition 1.6(b). In particular, we have $A = E \cong \mathrm{E}_{p^3}$, $\Omega_1(H) = A = H'$ and so $\mathrm{C}_G(A) = A$ implies that either $G = H$ (and G is isomorphic to any group given in Proposition 1.6(b)) or $|G : H| = p$ and G/A is isomorphic to the nonabelian group $\mathrm{S}(p^3)$ of order p^3 and exponent p. Suppose that we have $G/A \cong \mathrm{S}(p^3)$. Set $K = G'$ so that $A < K < H$ and $K/A = (G/A)'$ which gives $|K : A| = p$. It follows that K is the nonmetacyclic minimal nonabelian group of order p^4 given with

$$K = \left\langle k, l \;\middle|\; k^{p^2} = l^p = 1,\ [k, l] = m,\ m^p = [m, k] = [m, l] = 1 \right\rangle,$$

where $K' = \langle m \rangle \leqslant Z(G)$ and $\mho_1(K) = \langle k^p \rangle \leqslant Z(G)$ (since $p > 2$). It follows that $K' \times \mho_1(K) = Z(K) = Z(G) < A$ with $|A : Z(G)| = p$. Let $a \in A \setminus Z(G)$ so that $C_G(A) = A$ gives $C_G(a) = A$ and therefore the conjugate class of a in G has the size p^3. This implies that

$$\left|A \setminus Z(G)\right| = p^3 - p^2 \geqslant p^3,$$

which is a contradiction. Hence we must have $G = H$.

We have proved that if a p-group G satisfies the assumptions of Theorem 2.1 and if G possesses a normal elementary abelian subgroup of order p^3, then G is isomorphic to the groups defined in Proposition 1.6 (a) and (b). Conversely, it is clear that each group from Proposition 1.6 satisfies the assumptions of our theorem.

(ii) *Now assume that G has no normal elementary abelian subgroup of order* p^3.

(ii1) *First suppose in addition that G is nonmetacyclic.*

Let M be a minimal nonmetacyclic subgroup in G. Then Proposition 1.4 gives us the structure of M. In cases (b) and (c) of this proposition our subgroup M is nonabelian and possesses an abelian maximal subgroup (see also Proposition 1.5) but M is not minimal nonabelian, contrary to the property $(*)$. It follows that we must be in case (a) of Proposition 1.4 and so M is of order p^3 and exponent p. Suppose that M is nonabelian. Then we have $p > 2$ and $M \cong S(p^3)$ (the nonabelian group of order p^3 and exponent p) which is minimal nonabelian. In that case we have $M < G$ and let V be a subgroup of order p^4 which contains M. By Proposition 1.5, V has an abelian maximal subgroup but V is not minimal nonabelian, contrary to the property $(*)$. It follows that $M \cong E_{p^3}$. If $p > 2$, then by Proposition 1.8 our group G has also a normal elementary abelian subgroup of order p^3, contrary to our assumption.

We have proved that we must have $p = 2$ and G possesses a subgroup $M \cong E_8$. Let A be a maximal abelian subgroup of G which contains M. Let $B > A$ be a subgroup of G such that $|B : A| = 2$. By the property $(*)$, B is nonmetacyclic minimal nonabelian and so (Proposition 1.1) we have $M = \Omega_1(B)$. By our assumption we have $B < G$. Set $C = N_G(M)$ so that we have $B < C < G$. By part (i) of the proof, C is a special group of order 2^6 which is isomorphic to an S_2-subgroup of the simple group Sz(8). But then we have $M = \Omega_1(C)$ and so $N_G(M) > C$, a contradiction. We have proved that in case (ii) of the proof our group G cannot be nonmetacyclic.

(ii2) *Our group G is metacyclic.*

First we examine the case $p > 2$. Since G is neither abelian nor minimal nonabelian, we have $|G'| > p$ (see Proposition 1.2) and these are the groups of part (b) of our theorem. Conversely, let H be a nonabelian subgroup in a metacyclic p-group G (with $p > 2$) which possesses an abelian maximal subgroup. Then by Proposition 1.10, H is minimal nonabelian. Hence our groups in part (b) of our theorem have the property $(*)$.

We turn now to the case $p = 2$. Let G be a metacyclic 2-group which is neither abelian nor minimal nonabelian and which has the property $(*)$. Then we have (by Proposition 1.2) $|G'| > 2$. Let $\langle a \rangle$ be a cyclic normal subgroup of G such that $G/\langle a \rangle$ is cyclic. Let $b \in G$ be such that $G = \langle a, b \rangle$. We set $o(a) = 2^n$, where $n \geqslant 4$ since $|G : C_G(a)| > 2$. Indeed, if $|G : C_G(a)| \leqslant 2$, then G is either abelian or minimal nonabelian (by the property $(*)$), contrary to our assumption.

Assume that $a^b = a^{-1}v$, where $v \in \langle a^4 \rangle$. We consider the subgroup $S = \langle a^{2^{n-3}}, b \rangle$, where $o(a^{2^{n-3}}) = 8$. We have

$$\left(a^{2^{n-3}}\right)^b = \left(a^b\right)^{2^{n-3}} = \left(a^{-1}v\right)^{2^{n-3}} = a^{-2^{n-3}} v^{2^{n-3}},$$

where $o(v^{2^{n-3}}) \leqslant 2$. Hence b induces on $\langle a^{2^{n-3}} \rangle \cong C_8$ an involutory automorphism such that $S' = \langle a^{2^{n-2}} \rangle \cong C_4$ and b^2 centralizes $\langle a^{2^{n-3}} \rangle$. It follows that S is nonabelian but S is not minimal nonabelian (see Proposition 1.1). On the other hand, S has the abelian maximal subgroup $\langle a^{2^{n-3}}, b^2 \rangle$, a contradiction.

We have proved that we must have $a^b = av$, where $v \in \langle a^4 \rangle$. Hence G is ordinary metacyclic with respect to the subgroup $\langle a \rangle$ and so G is a group of part (b) of our theorem.

Conversely, let G be an ordinary metacyclic 2-group with $|G'| > 2$. Let U be a nonabelian subgroup in G which has an abelian maximal subgroup. By Proposition 1.11, U is also ordinary metacyclic and so by the same proposition, U is minimal nonabelian. We have proved that the groups in part (c) of our theorem have the property $(*)$. Our theorem is proved. $\square$

3. Solution of the problem 2331 in [4] of Y. Berkovich

Theorem 3.1. *Let G be a nonabelian p-group which is not minimal nonabelian. We assume that G has the following property*:

$(**)$ *Whenever X and Y are distinct minimal nonabelian subgroups of G and $x \in X \setminus Y$, $y \in Y \setminus X$, then $\langle x, y\rangle$ is also minimal nonabelian.*

Then G has also property $(*)$ (see Theorem 2.1) and we must have $p = 2$ and G is a special group of order 2^6 which is isomorphic to a Sylow 2-subgroup of the Suzuki simple group $\mathrm{Sz}(8)$.

Proof. Let G be a p-group satisfying the assumptions of our theorem.

(i) First we prove that whenever $B \leqslant G$ is nonabelian and A is an abelian maximal subgroup of B, then B is minimal nonabelian. (This is the property $(*)$ from Theorem 2.1.) Indeed, take an element $x \in B \setminus A$ and set $A_0 = \mathrm{C}_A(x)$ so that we have $x^p \in A_0$, $A_0 = \mathrm{Z}(B)$ and $B_0 = A_0\langle x\rangle$ is a maximal abelian subgroup of B. Note that $B_0 < B$ and let $B_1 \leqslant B$ be such that $B_1 > B_0$ and $|B_1 : B_0| = p$. Set $A_1 = B_1 \cap A$ so that $|A_1 : A_0| = p$ and $|B_1 : A_1| = p$. It follows that A_1 and B_0 are two distinct abelian maximal subgroups of the nonabelian group B_1. By Proposition 1.3, $B_1' = \langle z\rangle$ is of order p and $z \in A_0 = \mathrm{Z}(B)$. For each $y \in A_1 \setminus A_0$, we have $\langle [x, y]\rangle = \langle z\rangle$ and so $M = \langle x, y\rangle$ is minimal nonabelian (see Proposition 1.2). Let y_1 be any other element in $A_1 \setminus A_0$ so that $N = \langle x, y_1\rangle$ is minimal nonabelian and assume $M \neq N$. But then we have $y \in M \setminus N$ and $y_1 \in N \setminus M$ and so, by our property $(**)$, $\langle y, y_1\rangle$ must be minimal nonabelian, contrary to $y, y_1 \in A_1 \leqslant A$. Hence we must have $M = N$ and so $y_1 \in M$. It follows that $A_1 \setminus A_0 \leqslant M$ and so $\langle A_1 \setminus A_0\rangle = A_1 \leqslant M$ which gives that $B_1 = M$ is minimal nonabelian. Since x was an arbitrary element in $B \setminus A$, it follows that each minimal nonabelian subgroup M_0 of B contains $A_0 = \mathrm{Z}(B) = \mathrm{Z}(M_0) = \Phi(M_0)$ so that $|M_0| = |M| = p^2|A_0|$.

We claim that for each $x \in B \setminus A$, there is a unique minimal nonabelian subgroup $B_1 = M$ of B containing x. Indeed, assume that $B_2 \neq B_1$ is another minimal nonabelian subgroup of B containing x. Since $B_2 \geqslant A_0\langle x\rangle = B_0$, it follows $B_1 \cap B_2 = B_0$. We have $B_2 \cap A > A_0$ and let $v \in (B_2 \cap A) \setminus A_0$ so that $v \notin A_1$. On the other hand, $B_1 = \langle x, y\rangle$ with $y \in A_1 \setminus A_0$. Then we have $v \in B_2 \setminus B_1$ and $y \in B_1 \setminus B_2$ and so, by the property $(**)$, $\langle v, y\rangle$ must be minimal nonabelian, contrary to $\langle v, y\rangle \leqslant A$.

Suppose that $B \neq B_1$, where $B_1 = M = \langle x, y\rangle$ with $x \in B \setminus A$ and $y \in A_1 \setminus A_0$. Take an element $x_1 \in B \setminus (A \cup B_1)$. Let M_1 be a unique minimal nonabelian subgroup of B containing x_1 so that we have $M \neq M_1$. In that case we have $x \in M \setminus M_1$, $x_1 \in M_1 \setminus M$ and therefore (by the property $(**)$) $\langle x, x_1\rangle$ is minimal nonabelian. By the uniqueness of minimal nonabelian subgroups of B which contain a prescribed element in $B \setminus A$, we get $\langle x, x_1\rangle = M = M_1$, a contradiction. Hence our statement (i) is proved.

(ii) If A is a maximal normal abelian subgroup of G, then $\exp(G/A) = p$.

Indeed, let A be a maximal normal abelian subgroup of G. Suppose that there is a subgroup H/A of G/A such that $H/A \cong \mathrm{C}_{p^2}$. Let $h \in H \setminus A$ be such that $\langle h\rangle$ covers H/A and set $k = h^p$. Then, by (i), $B = A\langle k\rangle$ is minimal nonabelian. Let $A^* = \mathrm{C}_H(h) = \langle h\rangle \mathrm{C}_A(h)$ so that A^* is a maximal abelian subgroup of H. Let B^* be a subgroup of H such that $B^* > A^*$ and $|B^* : A^*| = p$. By (i), B^* is minimal nonabelian. Hence $k = h^p \in \Phi(B^*) = \mathrm{Z}(B^*)$ and so k centralizes $B^* \cap A$. This implies that $B^* < H$ and $B^* \cap A < A$. On the other hand, we have $\mathrm{C}_G(A) = A$ and so k does not centralize A. Let $a \in A \setminus \mathrm{C}_A(k)$ so that $a \in B \setminus B^*$. Also, $h \in B^* \setminus B$ and so (by the property $(**)$) $\langle a, h\rangle$ is minimal nonabelian. In this case, $h^p = k \in \mathrm{Z}(\langle a, h\rangle)$ and so k centralizes a, which is a contradiction. Our statement (ii) is proved.

(iii) We have $p = 2$ and whenever A is a maximal normal abelian subgroup of G, then A is of type $(4, 2, 2)$ and so G is nonmetacyclic.

Let A be any maximal normal abelian subgroup of G. By (i) and our assumption we have $|G/A| > p$. Let H/A be any subgroup of order p^2 in G/A. By (ii), we have $H/A \cong \mathrm{E}_{p^2}$.

By (i), for each of the $p+1$ subgroups H_i/A of order p in H/A, H_i is minimal nonabelian, $i = 1, 2, \ldots, p+1$. Let x, y be any elements in $H \setminus A$ such that $\langle x, y\rangle$ covers H/A. Set $H_1 = A\langle x\rangle$, $H_2 = A\langle y\rangle$ so that we have $H_1 \cap H_2 = A$. Since $x \in H_1 \setminus H_2$ and $y \in H_2 \setminus H_1$, it follows (by the property $(**)$) that $\langle x, y\rangle$ is minimal nonabelian. If $\Phi(H) = A$, then $\langle x, y\rangle = H$ is minimal nonabelian, contrary to the fact that H_1 is minimal nonabelian and $H_1 < H$. Thus $\Phi(H) < A$ which gives $\mathrm{d}(H) \geqslant 3$ and so, in particular, H is not metacyclic.

Now we have $|A : \Phi(H_1)| = p$, $|A : \Phi(H_2)| = p$ and $\Phi(H_1) \leqslant \Phi(H)$, $\Phi(H_2) \leqslant \Phi(H)$. This gives $\Phi(H_1) = \Phi(H_2) = \Phi(H)$, $|A : \Phi(H)| = p$ and so $\mathrm{d}(H) = 3$. Moreover, $\Phi(H_1) = \mathrm{Z}(H_1)$, $\Phi(H_2) = \mathrm{Z}(H_2)$ and so $\Phi(H) \leqslant \mathrm{Z}(H)$. By (i), H does not possess any abelian maximal subgroup and so we have $\Phi(H) = \mathrm{Z}(H)$.

Note that $\langle x, y\rangle$ and H_1 are two distinct minimal nonabelian subgroups of H. Suppose that $\langle x, y\rangle$ does not contain $\Phi(H) = \Phi(H_1)$ and let $f \in \Phi(H_1) \setminus \langle x, y\rangle$. Then we have $y \in \langle x, y\rangle \setminus H_1$ and $f \in H_1 \setminus \langle x, y\rangle$ so that (by the property $(**)$) $\langle f, y\rangle$ must be minimal nonabelian. This is a contradiction since $f \in \mathrm{Z}(H)$. Hence $\langle x, y\rangle > \Phi(H)$ and so $\langle x, y\rangle$ is a maximal subgroup of H.

We have proved that each maximal subgroup of H is minimal nonabelian and $\mathrm{d}(H) = 3$ so that H is nonmetacyclic. Also we have $|H| > p^4$. Indeed, if $|H| = p^4$, then Proposition 1.5 implies that H would have an abelian maximal subgroup, a contradiction. By Proposition 1.6(a), H is a special group of order 2^6 which is isomorphic to a Sylow 2-subgroup of the Suzuki simple group Sz(8). In particular, A is of type $(4, 2, 2)$ and so G is nonmetacyclic. Our statement (iii) is proved.

By (i), our group G satisfies the assumptions of Theorem 2.1. In addition, by (iii), we have $p = 2$ and G is nonmetacyclic. Hence Theorem 2.1 implies that G is isomorphic to a Sylow 2-subgroup of the Suzuki simple group Sz(8). Our theorem is proved. □

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